

The University of Chicago

UNIVERSAL QUADRATIC ZERO
FORMS IN FOUR VARIABLES

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1931

By
FRED WINCHELL SPARKS

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INTRODUCTION

In this paper universal quadratic zero forms in four variables are considered, that is, forms which represent all integers and which are zero for values of the variables not all zero. Professor Dickson¹ has proved that every such form may be reduced by a certain transformation to the form (i) $F = 2^e gaxy + gby^2 + cyz + gd(hz^2 + jzw + rw^2)$, where g and a are odd, a is prime to d , c is prime to g , and the g. c. d. of h , j , and r is unity. The necessary and sufficient conditions for the universality of F were completely established by Professor Dickson² for the cases $e = 0$, $e = 1$, d odd, c and j even; $e > 1$, d and r odd, c and j even. We seek the necessary and sufficient conditions for universality in all other cases. Due to previously established theorems³ regarding the form (i), we know that it is only necessary to investigate the solvability, with certain restrictions on y , of the congruence $F \equiv G \pmod{2^e}$, where G is arbitrary.

The method of procedure was to find, by use of various devices, solutions, with the prescribed restrictions on y , of the congruence $F \equiv G \pmod{2^n}$, where G is an arbitrary residue class of the modulus, and where n was allowed to run through all integers from 1 to q , and where q was usually 3 or 4. Where possible an induction from n to $n+1$ was es-

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1. Transactions of the American Mathematical Society, vol. 31, number 1, page 165, Theorem 2.
 2. Loc. cit., Theorems 8 to 13.
 3. Loc. cit., Lemma 3, page 173; Theorem 7, page 176.

tablished. In cases where solutions for the congruence could not be found, or where the induction could not be established, definite residue classes were exhibited with a proof that for these classes the congruence was not solvable. Various transformations were used, where possible, to simplify the form of F.

1. PRELIMINARY THEOREMS AND LEMMAS. Professor Dickson has proved¹ that every universal quadratic null form in three or more variables is equivalent to the form

$$(1) \quad F = 2^e g a x y + g b y^2 + c y z + g d \psi(z, w, \dots),$$

where g and a are odd, a is prime to d , c is prime to g , and the g. c. d. of the coefficients of ψ is unity.

For the four variable case we have

$$(2) \quad \psi = h z^2 + j z w + r w^2, \quad 1 = \text{g. c. d. of } h, j, r,$$

and the form (1) becomes

$$(3) \quad F = 2^e g a x y + g b y^2 + c y z + g d (h z^2 + j z w + r w^2).$$

The following theorems and lemmas have been established by Professor Dickson dealing with the form (3):

Lemma² I. If each of the congruences

$$(4) \quad F \equiv G \pmod{2^e}, \quad F \equiv G \pmod{g a}$$

has a solution x, y, z, w such that y has a fixed value 1 or an odd prime dividing no one of $g, a, d, h, N = j^2 - 4hr$, then $F = G$ is solvable.

Theorem³ I. For the form F defined by (3), let g and a be odd, a prime to d , c prime to g , and h prime to ga . Employ the abbreviations

$$(5) \quad N = j^2 - 4hr, \quad R = c^2 - 4g^2 d h b, \quad J = c^2 j^2 - N R.$$

We may assign to y a fixed value which is 1 or an odd prime such that $F \equiv G \pmod{g a y}$ is solvable for every G except in

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1. Transactions of the American Mathematical Society, vol. 31, number 1, page 165, Theorem 2.
 2. Loc. cit., page 173, Lemma 3.
 3. Loc. cit., page 176, Theorem 1.

the following cases:

- (6) $J \equiv 0 \pmod{p}$, $(N/p) = -1$; $N \equiv cj \equiv 0 \pmod{p}$
and either $R \equiv 0 \pmod{p}$ or $(R/p) = -1$,

where p is any prime dividing a but not g. In these cases
F is never universal.

Theorem II¹. Let $e \equiv 2$, $c = 2C$, $j = 2s$, d, r , and
 $\lambda = g^2 d^2 (rh - s^2)$ be odd. Write $M = \lambda g^2 d b r - g^2 d^2 r^2 C^2$.
Then F is universal, if and only if we exclude cases (6)
and the following:

$M \equiv 0 \pmod{4}$; $\lambda \equiv 3 \pmod{8}$, $e \equiv 4$; $\lambda \equiv M \equiv 1 \pmod{4}$,
 $e \equiv 3$; $\lambda \equiv 1 \pmod{4}$, $M \equiv 1 + \lambda \pmod{8}$, $e = 4$.

We shall now prove

Lemma II. If $F \equiv G \pmod{2^e}$ is solvable only when y is
divisible by 2^s , F is universal, except in cases (6), if
and only if $F \equiv G \pmod{2^{e+s}}$ is solvable.

If F is universal, $F \equiv G \pmod{2^{e+s}}$ is solvable. Suppose $F \equiv G \pmod{2^{e+s}}$ is solvable with $y = 2^s y'$, where y' satisfies the conditions of Theorem I. Then F is replaced by a form derived from F by replacing e by $e+s$, b by $2^s b$, c by $2^s c$, and y by y' . M is not changed and R and J are each multiplied by 2^s , and hence conditions (6) are not altered. Thus even values of y are not excluded.

We shall also need

Lemma III. If, for a given integer n, the congruences
 $F \equiv G \pmod{2^n, 2^{n+1}}$ are solvable where G is any integer,
and if $F \equiv K \pmod{2^m}$ is solvable for all integers $m > n$,

1. Transactions of the A. M. S., vol 31, number 1, page 180, theorem 10.

where K is any integer not a multiple of 4, then $F \equiv G \pmod{2^m}$ is solvable, where G is arbitrary.

By the first hypothesis we have $4F \equiv 4G \pmod{2^{n+2}, 2^{n+3}}$. But $4F(x, y, z) = F(X, Y, Z)$ where $X = 2x$, $Y = 2y$, $Z = 2z$. Hence $F \equiv G \pmod{2^{n+2}, 2^{n+3}}$ is solvable with G arbitrary. We next suppose $F \equiv G \pmod{2^{n+2k}, 2^{n+2k+1}}$ solvable, with G arbitrary. By a repetition of the above argument we have $F \equiv G \pmod{2^{n+2(k+1)}, 2^{n+2(k+1)+1}}$ solvable, and hence the induction on k is complete. Therefore, since m is such an exponent, we have $F \equiv G \pmod{2^m}$ solvable, where G is arbitrary.

Professor Dickson has determined the necessary and sufficient conditions for the universality of F for the cases $e = 0$; $e = 1$, d odd, c and j even; $e = 1$, d and c even; $e > 1$, d and r odd, c and j even¹. It is the purpose of this paper to determine the necessary and sufficient conditions for the universality for all other cases. In view of Lemma I and Theorem I it is necessary to consider only the conditions under which the first of the congruences of (4) is solvable.

Let

$$(7) \quad A = ga, \quad B = gb, \quad H = gdh, \quad 2^u S = gdj, \quad 2^v L = gdr,$$

where S and L are odd. Then the form (3) becomes

$$(8) \quad F = 2^e Axy + By^2 + cyz + Hz^2 + 2^u Szw + 2^v Lw^2.$$

2. THE CASES $u = 0$, $v > 0$; $u = 0$, $v = 0$. We first consider $u = 0$, $v > 0$. If z is odd then $t = Szw + 2^v Lw^2$ runs

1. Dickson, "Universal Quadratic Forms", Transactions of the A. M. S., vol. 31, no. 1, Theorems 8-13.

through all the residues of the modulus 2^e with w , for if $t_1 \equiv t_2 \pmod{2^e}$ for two values w' and w'' of w , we have $(w' - w'') [Sz + 2^v L(w' + w'')] \equiv 0 \pmod{2^e}$. For odd values of z the second factor is always odd, and hence $w' \equiv w'' \pmod{2^e}$. If we assign to y and z fixed odd values we have $F \equiv K + t \pmod{2^e}$, where K is a constant. Hence $F \equiv G \pmod{2^e}$ is always solvable where G is arbitrary, and we have

Theorem III. F is universal when $u = 0$, $v > 0$ if and only if cases (6) are excluded.

If $u = 0$ and $v = 0$, we first assign to y and z fixed odd values y' and z' respectively and we obtain $F \equiv K' + t \pmod{2^e}$, where $K' = By'^2 + cy'z' + Hz'^2$, and where t has the value in the above paragraph with $v = 0$. If for two different odd values w' and w'' of w we have $t' \equiv t'' \pmod{2^e}$ it follows that $(w' - w'') [S + L(w' + w'')] \equiv 0 \pmod{2^e}$. Since the second factor is always odd we have $w' \equiv w'' \pmod{2^e}$ and this contradicts the hypothesis. Hence if we assign to w , 2^{e-1} different odd values, t will run through all of the even residues of the modulus 2^e . If all three of the integers B , c and H are odd, or if two of them are even and the other is odd, K' is odd and $F \equiv G \pmod{2^e}$ is always solvable when G is odd. If, on the other hand, B , c , and H are all even, or if two of them are odd while the other is even, K' is even and $F \equiv G \pmod{2^e}$ is always solvable when G is even.

When y is replaced by $2Y$ the form F is replaced by $F' = 2^{e+1}AxY + 4BY + 2cYz + Hz^2 + Szw + Lw^2$.

We now assign to Y and z fixed odd values y' and z' respectively, and we have $F' \equiv K'' + t \pmod{2^{e+1}}$, where $K'' = 4By'^2 + 2cy'z' + Hz'^2$ is even or odd with H . Hence $F' \equiv G \pmod{2^{e+1}}$ is solvable when G and H are odd, or when G and H are even. Thus when either B or c is odd while the other is even we may apply Lemma II and F is universal, subject to conditions (6).

If B and c are both even or both odd we need the even residues of 2^e when H is odd, and the odd residues when H is even. In the first case $F \equiv y(B + cz) + z + w(z+1) \pmod{2}$, hence F is odd when z is odd. In the second case $F \equiv y(B + cz) + w(z + 1) \pmod{2}$ and F is odd only when z is even. Hence in either case we obtain the desired residues only when z is even. Let $z = 2z'$, with z' arbitrary. Since L is odd we may confine our attention to the form $TLF = 2^e ALxy + BLy^2 + 2cLy z' + Tz'^2 + W^2$, where $T = 4HL - S^2$, $W = Sz' + Lw$. Let $M = BLT - c^2L^2$ and $Z = cLy + Tz'$, and we have $TLF = 2^e ATLxy + My^2 + Z^2 + TW^2$. Since T and L are odd and our moduli are powers of 2 we may take y, Z , and W new variables in place of y, z' , and w .

When $e = 1$ we take y odd, $w = 0$, $z = 0$ then 1 and we have $TLF \equiv M, M+1 \pmod{2}$. Now let $e = 2$. Evidently $T \equiv 3 \pmod{4}$, and, since B and c are both even or both odd, M is even. When H is odd we need to show that the congruence

$$(9) \quad TLF \equiv P \pmod{2^e}; \quad P = TLG$$

is solvable, with the proper restrictions on y , when P is

even. In case $M \equiv 0 \pmod{4}$, $TLF \equiv Z^2 + 3W^2 \not\equiv 2 \pmod{4}$, and hence F is not universal. If $M \equiv 2 \pmod{4}$ and $e = 2$, let Z and W be odd and we have $TLF \equiv 2 \pmod{4}$. Now let $y = 2y'$, where y' is odd, and we have $TLF \equiv 0, 4 \pmod{8}$ by choice of Z and W . If H is even we must show that the congruence (9) is solvable when P is odd. If we let $Z = 0$, $W = 1$, then $Z = 1$, $W = 0$ we have $TLF \equiv M + 3, M + 1 \pmod{4}$, and these are the two residues required.

If $e = 3$ and H is odd, $T \equiv 3 \pmod{8}$, and in view of the above paragraph, $M \equiv 2 \pmod{4}$, and we require P in the congruence (9) to be even. Hence $k = Z^2 + 3W^2$ is congruent to 0, 4 modulo 8, and to 0, 4, 12 modulo 16, by choice of Z and W . When y is odd $TLF \equiv M, M+4 \pmod{8}$ is solvable, and each of these residues is twice an odd. We now take $y = 2y'$, where y' is odd, and we have $TLF \equiv 4My'^2 + k \equiv 4M, 4M + 4 \pmod{8}$, and $TLF \equiv 4M, 4M + 4, 4M + 12 \pmod{16}$. Let $y = 4y'$, with y' odd, $Z = W = 0$ then $Z = 4$, $W = 0$ and we have $TLF \equiv 0, 16 \pmod{32}$. We have thus shown that TLF represents the two residue classes of 8 which are twice an odd, and, since $M \equiv 2 \pmod{4}$, F represents the residue classes 8, 12 and 4 of 16 and 0, 16 of 32. All even integers are included in the above residue classes, and the conditions of Lemma II are satisfied. Hence when B and c are both even or both odd, $e = 3$, $M \equiv 2 \pmod{4}$ and H is odd, F is universal, subject to conditions (6)

Since $k \equiv 0 \pmod{4}$ when Z and W are both even or both odd, TLF is twice an odd only when y is odd, and hence $y^2 \equiv 1$

$(\text{mod } 4)$ and $My^2 \equiv 2 \pmod{16}$, and since k represents only three even residues of 16, TLF represents only three residues of 16 which are twice an odd, and hence F is not universal when $e > 3$.

If H is even, T is congruent to 7, modulo 8, and we need to show that (9) is solvable when P is odd. $TLF \equiv My^{2+k} \pmod{8}$ where $k \equiv Z^2 + 7W^2 \pmod{8}$. By choice of Z and W we may obtain $k \equiv 1, 3, 5, 7 \pmod{8}$ and hence TLF represents a complete set of odd residues, $M+1, M+3, M+5, M+7$ of 8.

To proceed by induction from $m \leq 3$ to $m+1$ we let $x = 0$ and $My^2 + Z^2 + TW^2 = K + 2^m Q$. Then $My^2 + (Z + 2^{m-1} \xi)^2 + T(W + 2^{m-1} \omega)^2 \equiv K \pmod{2^{m+1}}$ if $Q + 2\xi + TW\omega \equiv 0 \pmod{2}$. The last congruence has solutions when H is even since Z and W are not both even.

This completes proof of
Theorem IV. We employ notation (7). F is universal when $u = v = 0$ if $e = 1$. If $e > 1$ F is universal unless H is odd. If H is odd and $e > 3$ F is never universal, but when $e = 2$ or 3, F is universal if and only if $M \equiv 2 \pmod{4}$, where $M = BLT - c^2 L^2$. Universality in each case is qualified by excepting conditions (6).

3. THE CASE $u > v$, c EVEN, H ODD. The case in which r and d are odd has been disposed of by Professor Dickson¹, and hence when r is odd, we need only to consider d even.

1. Dickson, "Universal Quadratic Forms," Transactions of the A. M. S., vol 31, no. 1.

Therefore in all events we have $v > 0$. Let $c = 2C$ and $u - v = \xi$.

Since L is odd, $F \equiv G \pmod{2^e}$ is solvable if $LF \equiv LG \pmod{2^e}$ is solvable, where G is arbitrary and where

$$(10) \quad \begin{aligned} F'' = LF &= 2^e ALxy + BLy^2 + 2CLyz + Tz^2 + 2^v W^2, \\ T &= HL - 2^e S^2, \quad W = Lw + 2^{e-1} Sz, \quad e = v + 2\xi - 2 > 0. \end{aligned}$$

Since H is odd, T is odd and we may confine our attention to

$$(11) \quad \begin{aligned} F' = TLF &= 2^e ATLxy + My^2 + Z^2 + 2^v TW^2, \\ M &= BTL - C^2 L^2, \quad Z = Tz + CLy. \end{aligned}$$

T and L are odd and our moduli are powers of 2, Hence we may take y , Z , and W as our new variables.

The case in which $e = 1$ has been investigated by Professor Dickson¹, hence we will deal with $e \geq 2$. If $v > 1$, $F' \equiv My^2 + Z^2 \pmod{4}$. If M is odd, F' represents the even residues 0 and $M+1$ of 4, which differ only when $M \equiv 1 \pmod{4}$, and then we are able to obtain only the single odd residue 1. If M is even F' is even only when Z is even, and then we have $F' \equiv 0, M \pmod{4}$ and these differ only when $M \equiv 2 \pmod{4}$.

(i) The case $v > 1$. Let $e = 2$, $M \equiv 2 \pmod{4}$. If $v > 1$ F' is never odd unless Z is odd and then $F' \equiv 2y^2 + 1 \pmod{4}$. Obviously $F' \not\equiv 1 \pmod{4}$ unless y is even, but if $v > 2$, F' is never congruent to 5, modulo 8. When $v = 2$ we have $F' \equiv 2, 3 \pmod{4}$ with y odd, $Z = 0, 1$. Let $y = 2y'$, y' odd, and we have $F' \equiv 0, 1 \pmod{8}$ with $W = 0$, and $F' \equiv 4, 5 \pmod{8}$ with $W = 1$. Hence by Lemma II, F is universal when

1. Transactions of the A. M. S., vol. 31, no. 1, page 177, Theorem 9.

$e = 2$ if and only if $M \equiv 2 \pmod{4}$ and $v = 2$.

If $e = 3$, we first consider $v \equiv 3$, and we have $F' \equiv My^2 + Z^2 \pmod{8}$. Since M is even, F' is odd only when Z is odd and hence F' represents only two odd residues, 1 and $M+1$, of 8. Therefore $v = 2$ and $F' \equiv My^2 + Z^2 + 4TW^2 \pmod{8}$. Assign odd values to y and Z and let $W = 0$ then 1, and we have $F' \equiv M+1, M+5 \pmod{8}$. Since $M \equiv 2 \pmod{4}$ these residues are congruent to 3 and 7, modulo 8. Let $y = 2y'$, y' odd, $W = 0$, $Z = 1$ then 3 and we have $F' \equiv 9, 1 \pmod{16}$. Let $W = 1$, y' and Z as before and we have $F' \equiv 13$ and 5, $\pmod{16}$. All odd integers are included in the above residue classes and the conditions of Lemma II are satisfied.

If y is odd, $W = 0$, $Z = 0$ then 2 we have $F' \equiv M, M+4 \pmod{8}$, and these two residue classes include all even integers which are twice an odd. If we take $y = 2y'$, y' odd, and $W = 0$, we have $F' \equiv 0, 4 \pmod{8}$ by choice of Z , but in order to apply Lemma II we must have $F' \equiv 0, 4, 8, 12 \pmod{16}$. Since $4My'^2 \equiv 8 \pmod{16}$ we have $F' \equiv 8, 12 \pmod{16}$ when $W = 0$, by choice of Z . Let $W = 1$, then $4TW^2 \equiv 4T \equiv 4$ or $12 \pmod{16}$ according as T is congruent to 1 or 3, modulo 4. In the first case F represents the additional residue 0 of 16, and in the second case, the additional residue 4. Let $y = 4y'$, y' odd, and $F \equiv Z^2 + 4TW^2 \pmod{32}$. When $T \equiv 1 \pmod{4}$ we need the residue 4 of 16, or the residues 4 and 20, of 32. We have $F' \equiv 4 \pmod{16, 32}$ with $Z = 2$, $W = 0$. If $4T \equiv 4 \pmod{32}$ we have $F' \equiv 20$

(mod 32) with $Z = 2, W = 2$. If $4T \equiv 20 \pmod{32}$ we have $F' \equiv 20 \pmod{32}$ with $Z = 0, W = 1$. When $T \equiv 3 \pmod{4}$, $F' \equiv 0, 16 \pmod{32}$ with $W = 0, Z = 0$ then 4.

In case $e > 3$, we have $F' \equiv My^2 + Z^2 + 4TW^2 \pmod{16}$. Evidently F' represents integers which are twice an odd only when y is odd and Z is even. Let $M = 2M'$, M' odd, $Z = 2Z'$, then $F' \equiv 2p \pmod{16}$ where $p = M'y^2 + 2Z'^2 + 2TW^2$, and hence F' represents all residues of 16 which are twice an odd if and only if $p \equiv k \pmod{8}$ is solvable with k odd. Obviously p is odd only when y' is odd, and it represents only the residues $M', M' + 2, M' + 2T$, and $M' + 2 + 2T$, of 8. We lack the residue $M + 6$ of 8 if $T \equiv 1 \pmod{4}$, and the residue $M + 4$ of 8 if $T \equiv 3 \pmod{4}$. Hence when $e > 3$, $v > 1$, F is never universal.

Thus we have

Theorem V. Employ notation (7). If $e \geq 2, v \geq 2$, and $u > v$, F is universal if and only if $e = 2$ or $3, v = 2$ and $M \equiv 2 \pmod{4}$, where $M = BTL - C^2L^2, c = 2C$. Universality is qualified by excluding cases (6).

(ii) The case $v = 1$. Let $e = 2$, and take y odd. By choice of Z and W we have $F' \equiv M, M+1, M+1+2T, M+2T \pmod{4}$, and this constitutes a complete set of residues of 4. Hence F' is universal except for cases (6).

We now consider $e \geq 3$, and first let M be even. In case $M \equiv 0 \pmod{8}$, we have $F' \equiv Z^2 + 2TW^2 \pmod{8}$. Since $2TW^2$ is congruent only to 0 and $2T$, modulo 8, F' represents, at most, six of the odd residues of 8, and therefore is

not universal. Evidently F' is odd only when Z is odd, and then $F' \equiv My^2 + 1 + 2TW^2 \equiv 1, 1 + 2T, M + 1, M + 1 + 2T \pmod{8}$. If $M \not\equiv 0 \pmod{8}$ the above residues differ if and only if $M \not\equiv \pm 2T \pmod{8}$, since neither M nor T is congruent to 0, modulo 8. Hence when M is even the only possibility for universality is $M \equiv 4 \pmod{8}$.

Since under these conditions $F' \equiv Z^2 + 2TW^2 \pmod{4}$, F' represents integers which are twice an odd only when Z is even and W is odd. If $e > 3$ and Z and W are subject to these restrictions, we have $F' \equiv M + 2T, M + 4 + 2T, 2T$, and $4 + 2T \pmod{16}$. Since $M \equiv 4 \pmod{8}$, either the first of these residues is congruent to the last or the second is congruent to the third, and therefore F' is never universal when $e > 3$. When $e = 3$, we take y odd, Z even, and let W be odd and then even, and we have $F' \equiv M + 2T, M + 4 + 2T, M$, $M + 4 \pmod{8}$. Since $M \equiv 4 \pmod{8}$, this constitutes a complete set of even residues of 8.

When y and Z are odd and we let W be even, then odd, we have $F' \equiv 5, 5 + 2T \pmod{8}$. If $y = 2y'$, y' odd, we have $F' \equiv Z^2 + 2TW^2 \pmod{16}$. If we take W even, then odd, we have by suitable choice of Z , $F' \equiv 1, 9, 1+2T, 9+2T \pmod{16}$. All odd integers are included in the above residue classes, since, when $2T \equiv 2 \pmod{8}$, the above residues become 5 and 7 of 8, and 1, 9, 3 and 11 of 16, and if $2T \equiv 6 \pmod{8}$ they become 5 and 3 of 8, and 1, 9, 7 and 15 of 16. The conditions of Lemma II are satisfied, and hence F' is universal, except for conditions (6) when

$e = 3$ and $M \equiv 4 \pmod{8}$.

For odd values of M , F' is even only when y and Z are both even or both odd. If W is even, $2TW^2$ is congruent either to 0 or 8, modulo 16. If x is even or $e > 3$, $2TW^2 \equiv 0 \pmod{16}$, and if y and Z are odd we have $F' \equiv M+1, M+9, 9M+1, 9M+9 \pmod{16}$. If x is odd and $e = 3$ the first term of F' only serves to add 8 to each of the above residues, and this would only permute them. Likewise if $2TW^2 \equiv 8 \pmod{16}$ the above residues are unchanged. Evidently the first of these residues is congruent to the last, and the second is congruent to the third, modulo 16 in each case. Hence when y and Z are odd and W is even we only have

(12) $F' \equiv M+1, M+9 \pmod{16}$.

If y and Z are both even, W odd, and $e \leq 3$

(13) $F' \equiv 2T, 4M+2T, 4+2T, 4M+4+2T \pmod{16}$.

If $M \equiv 1 \pmod{4}$, we have $y' \equiv 2 \pmod{4}$ if and only if y and Z are odd and W is even, or if y and Z are even and W is odd. Hence, under these conditions, the residues (12) and (13) include all integers which are twice an odd represented by F' . If $M \equiv 1 \pmod{8}$, $2T \equiv 2 \pmod{16}$, the residues (12) are 2 and 10 and those in (13) are 2, 6, 6, 10. If $2T \equiv 10 \pmod{16}$, the residues (13) are 10, 14, 14, 2. In the first case, the congruence $F' \equiv 14 \pmod{16}$ is not solvable, and in the second, $F' \equiv 6 \pmod{16}$ is not solvable.

In case $M \equiv 5 \pmod{8}$ and $2T \equiv 6 \pmod{16}$ the residues in the congruences (12) become 6 and 14, and the four residues in (13) become 6, 10, 10, 14. Hence F' fails to rep-

represent 2, modulo 16. If $2T \equiv 14 \pmod{16}$, the residues in (13) are 14, 2, 2, 6, and hence F' fails to represent 10, modulo 16.

When $M \equiv 3 \pmod{4}$, F' is twice an odd only when W is odd, since $F' \equiv 3y^2 + Z^2 + 2W^2 \equiv 2 \pmod{4}$ is not solvable with W even. Under these conditions $2TW^2 \equiv 2T \pmod{16}$, and if $F' \equiv 8 + 2T \pmod{16}$ is solvable, we must have $My^2 + Z^2 \equiv 8 \pmod{16}$. Since $M \equiv 3 \pmod{8}$, the congruence $My^2 + Z^2 \equiv 0 \pmod{8}$ is satisfied only by values of y and Z which are multiples of 4 or both twice an odd, and then we have $My^2 + Z^2 \equiv 0 \pmod{16}$. Hence F' is never congruent to $8 + 2T \pmod{16}$ and is not universal.

We will now prove that, except for the conditions on M and T treated in the above paragraphs, $F' \equiv K \pmod{2^n}$ is always solvable if $K = TLG$ is any odd integer, or any even integer which is the double of an odd.

If y and Z are odd and $W = 0$, F' represents the residues $M+1$ of 8, and $M+1$ and $M+9$ of 16. If y is the double of an odd, W odd, and $Z = 2$, we have $F' \equiv 2T \pmod{8}$ and $F' \equiv 4M+4+2T \pmod{16}$, and if y is four times an odd, $Z=0$, and W is odd, we have $F' \equiv 2T \pmod{16}$. These residue classes include all integers which are the double of an odd if either $M \equiv 1 \pmod{8}$ with $T \equiv 3 \pmod{4}$, or $M \equiv 5 \pmod{8}$ with $T \equiv 1 \pmod{4}$.

For values of $M \equiv 7 \pmod{8}$ we have $F' \equiv 2T \pmod{8}$, and $F' \equiv 2T, 8+2T \pmod{16}$ with y, Z and W all odd. If $y = 2y'$, y' odd, $Z = 0$, $W = 1$, we have $F' \equiv 4 + 2T \pmod{8}$ and $F' \equiv 12 + 2T \pmod{16}$. If $y = 4y'$, y' odd, $Z = 2$, $W = 1$,

$F' \equiv 4+2T \pmod{16}$. Hence F' represents all residues of 8 and 16 which are the double of an odd.

To proceed by induction from $m \geq 4$ to $m+1$, we let $x=0$, and $F' = My^2 + Z^2 + 2TW^2 = K + 2^m Q$. The substitution of $Z = Z + 2^{m-1} \xi$, $W = W + 2^{m-2} \omega$ into the above value of F' yields $F' = K + 2^m(Q + Z\xi + TW\omega) + 2^{2m-2} \xi^2 + 2^{2m-3} T\omega^2$. Hence $F \equiv K \pmod{2^{m+1}}$ has solutions ξ, ω if either Z or W is odd and $m \geq 4$. Hence we have $F' \equiv K \pmod{2^n}$ always solvable when K is twice an odd, and the conditions of Lemma II are satisfied.

The odd residues $M, M+2T, M+4, M+4+2T$ of 8 represented by F' are obtained by assigning an odd value to y and by allowing Z to take the values 0, 2 and W to take the values 0, 1 for each value of Z . To proceed by induction we let $x = 0$ and $F' = My^2 + Z^2 + 2TW^2 = K + 2^m Q$. Substitute $y = y + 2^{n-1} \eta$ into F' and we have $F' \equiv K + 2^m(Q + My\eta) + 2^{2m-2} M\eta^2 \equiv K \pmod{2^{m+1}}$ if y is odd and $m \geq 3$.

It is now necessary to prove that when M is odd the congruence $F' \equiv K \pmod{2^n}$ is solvable where K is any multiple of 4. We let $x = W = 0$. Then $F' = My^2 + Z^2$ is congruent to 0, 4 modulo 8, or to 0, 4, 8 (modulo 16), where $y = 2^s y'$, with y' any assigned odd, and where Z is suitably chosen. If $M \equiv 1 \pmod{4}$, we let $y = 2y'$, y' odd, $Z = 0$, $W = 2$ and we have $F' \equiv 12 \pmod{16}$. If $M \equiv 3 \pmod{4}$ we let $Z = W = 0$ and then $F' \equiv 12 \pmod{16}$. Hence the conditions of Lemma III are satisfied.

This completes the proof of

Theorem VI. We employ the notations (7). If $u > v$, c is even, H is odd, and $v = 1$, F is universal if and only if the conditions (6) are excluded, and if the following conditions hold : $e = 2$; $e = 3, M \equiv 4 \pmod{8}$; $e \equiv 3, M \equiv 1 \pmod{8}, T \equiv 3 \pmod{4}$; $e \equiv 3, M \equiv 5 \pmod{8}, T \equiv 1 \pmod{4}$; $e \equiv 3, M \equiv 7 \pmod{8}$; where $M = BTL - C^2L^2, T = HL - 2\sigma S^2, \sigma = v + 2\varepsilon - 2, \varepsilon = u - v.$

4. THE CASE c EVEN, H EVEN, $u > v$. As in section 3, it is necessary to deal only with cases in which $v > 0$. Let $c = 2C, H = 2H'$. Since F shall represent odd integers, B is odd, and then $F \equiv B \pmod{2}$ is solvable and Theorem I applies. Evidently F is even only when y is even, and if $y = 2y', F = 2\varphi$, where

$$\varphi = 2^e Axy'^2 + 2By'^3 + 2Cy'z + H'z^3 + 2^{u-1}Szw + 2^{v-1}Lw^2.$$

F represents every even integer $2K$ if and only if φ represents every integer K . If $e = 1$ and $v > 1$, φ is never universal if H' is even, since in that case φ represents only multiples of 2. If H' is odd, $\varphi \equiv H', 0 \pmod{2}$ is solvable. If $v = 1$, $\varphi \equiv 0, L \pmod{2}$ is solvable. Hence in either case Theorem I applies and we have

Theorem VII. We employ notations (7). If c and H are even, $u > v \equiv 1$, and $e = 1$, F is universal if and only if B is odd, and either $v = 1$, or $H \equiv 2 \pmod{4}$, and if we exclude conditions (6).

If $e > 1$, we employ the notation of the preceeding section.

Since H is even, T is even, and it is necessary to study the form (10) shown on page 8. Since F'' shall represent odd integers, B is odd, and then it is evident that F'' is even only when y is even, and then if $v > 1$, $T \equiv 0 \pmod{4}$, F'' is not universal, since it represents only multiples of 4. Let $C = 2C'$, $T = 2T'$, $y = 2y'$, and suppose $v > 3$. If $x = 0$, $F'' \equiv 4BLy'^2 + 8C'Ly'z + 2T'z^2 \pmod{16}$, hence F'' represents only the residues $4BL + 8C'L + 2T'$, $4BL$, and $4BL + 8T'$ with y' odd, and the residues $2T'$, 0 , and $8T'$ with y' even, modulo 16, in both cases. If x is odd the above residues would be unchanged unless $e = 2$, and then the only effect would be the addition of 8 to the first set, and this contributes only one additional. Thus F'' represents at most seven of the eight even residues of 16. If $v = 1$ and $T \equiv 4 \pmod{8}$ and C is odd, $F'' \equiv 2 \pmod{8}$ is solvable only when $y \equiv 0 \pmod{4}$ and z is even, and under these conditions $F'' \not\equiv 10 \pmod{16}$. If C is even, $T \equiv 0 \pmod{16}$ and $v = 1$, F'' again fails to represent the residue 10 of 16. F'' represents odd integers only when y is odd, and then if $e \leq 2$, $v > 1$, $T \equiv 2 \pmod{4}$ and C is odd, F'' represents only the odd residue BL of 4. Hence F'' is never universal in the following cases: $T \equiv 0 \pmod{4}$, $v > 1$; C even, $v > 3$; C odd, $T \equiv 4 \pmod{8}$, $v = 1$; C odd, $T \equiv 2 \pmod{4}$, $v > 1$; C even, $T \equiv 0 \pmod{16}$, $v = 1$.

We first consider $e = 2$. If $v = 1$, take y odd, $z = 0$ and we have $F'' \equiv BL, BL+2 \pmod{4}$ by choice of W , and these residue classes include all odd integers. We now let $y=2y'$,

take y' odd and $z = 0$, and we have, by choice of W

$$(i) \quad F'' \equiv 4BL, 4BL + 2 \pmod{8}.$$

If C is odd with y' and z odd we have the congruences

$$(ii) \quad F'' \equiv T, T+2 \pmod{8}$$

solvable, and if C is even

$$(iii) \quad F'' \equiv 4BL + T, 4BL + T + 2 \pmod{8}$$

are solvable. When $y = 4y'$, $T \equiv 2 \pmod{4}$ with C even or odd, or $T \equiv 8 \pmod{16}$ with C even

$$(iv) \quad F'' \equiv 0, 2, 8, 10 \pmod{16}$$

are solvable. Hence when C is odd and $T \equiv 0 \pmod{8}$ we have the residues 4, 6, 0, 2 of 8, from the congruences (i) and (ii).

If $T \equiv 2 \pmod{4}$ we have the residues 4 and 6 of 8 from (i) and 0, 2, 8, 10 of 16 from (iv). If C is even, $T \equiv 4 \pmod{8}$

we see from (i) and (iii) that F'' represents the residues

4, 6, 0 and 2 of 8, and if $T \equiv 8 \pmod{16}$ with C even we

have the residues 4 and 6 of 8 from (i), and 0, 2, 8 and 10

of 16 from (iv). All integers are included in the above residue classes and the conditions of Lemma II are satisfied.

In case $v > 1$, C even, $T \equiv 2 \pmod{4}$, we let $T = 2T'$ with T' odd. We have $F'' \equiv BL, BL + 2T' \pmod{4}$, with $W = 0$.

If $y = 2y'$, y' odd, let $W = 0$, $z = 0$ then 1 and we have

$F'' \equiv 4BL, 4BL + 2T' \pmod{8}$. If $v = 2$ take W odd, $z = 0$

then 1 and then $F'' \equiv 4BL + 4, 4BL + 4 + 2T' \pmod{8}$. If

$v = 3$, let $y = 4y'$ and the congruences $F'' \equiv 0, 2T', 8,$

$8 + 2T' \pmod{16}$ are solvable. All integers are included

in the above residue classes, and the conditions of Lemma II

are satisfied. Hence we have

Theorem VIII. Employ notations (7) and let $c = 2C$. If we except conditions (6), F is universal when H is even, $u > v$, and $e = 2$ if and only if B is odd and the following conditions hold: C odd, $v = 1$, $T \equiv 2 \pmod{4}$ and $T \equiv 0 \pmod{8}$; C even, $v = 1$, $T \equiv 2 \pmod{4}$, $T \equiv 4 \pmod{8}$, $T \equiv 8 \pmod{16}$; C even, $v = 2$ or 3 , $T \equiv 2 \pmod{4}$; where $T = HL - 2\sigma S^2$, $\sigma = v + 2\xi - 2$, $\xi = u - v$.

When $e \leq 3$, we first consider $T \equiv 2 \pmod{4}$. Obviously F'' is odd only when y is odd, and, under these conditions, if C is even and $v = 1$, $t = 2CLyz + Tz^2 + 2^v W^2 \equiv 0 \pmod{4}$ is solvable only with z and W both even or both odd, and then if either $C \equiv 2 \pmod{4}$, $T \equiv 2 \pmod{8}$ or $C \equiv 0 \pmod{4}$, $T \equiv 6 \pmod{8}$, t is never congruent to 4, modulo 8, and hence F'' never represents a residue of 8 of the type $BL + 4$. On the other hand t is congruent to 2, modulo 4, only when one of z , W is odd and the other is even, and then if either $C \equiv 2 \pmod{4}$, $T \equiv 6 \pmod{8}$ or $C \equiv 0 \pmod{4}$, $T \equiv 2 \pmod{8}$, t is never congruent to 6, modulo 8. Hence F'' fails to represent $BL + 6$. Hence F'' is never universal when C is even, $T \equiv 2 \pmod{4}$, and $v = 1$. If $v > 2$ we have $F'' \equiv BLy^2 + 2CLyz + Tz^2 \not\equiv BL + 4 \pmod{8}$.

If C is odd, an odd prime value w may be assigned to y , and then z may be determined so that $2CLw^2 \equiv 2, 6 \pmod{8}$. Obviously such a value of z is odd, and then, if $v = 1$, we may have by choice of W , $F'' \equiv BL + 2 + T$, $BL + 6 + T$, $BL + 4 + T$, $BL + T \pmod{8}$, and this constitutes a complete set of odd residues.

If $v = 2$, $t \equiv 0 \pmod{4}$ if C and y are odd, regardless

of the value of z , and hence F'' fails to represent the residues $BL+2$ and $BL+6$ of 8.

To proceed by induction we let $x = 0$ and $F'' = k + 2^n Q$. Then $F''(0, y+2^{n-1}\eta, z+2^{n-1}\xi, w) \equiv k \pmod{2^{n+1}}$ has solutions η, ξ if $Q + BLy\eta + CLz\eta + CLy\xi \equiv 0 \pmod{2}$ is solvable. The latter congruence always has solutions since y is odd. Hence, when C is odd and $T \equiv 2 \pmod{4}$, $F'' \equiv k \pmod{2^e}$ is solvable when $v = 1$, where $k = LG$ and G is an arbitrary odd.

Since F'' is even only when y is even, we take $y = 2y'$. Evidently $T = 2T'$ where T' is odd. Under these conditions, $F'' \equiv 2K \pmod{2^e}$ is solvable if and only if $2\mathcal{O} \equiv 2P \pmod{2^e}$ is solvable, where P is any arbitrary integer and

$$(14) \quad \begin{aligned} \mathcal{O} &= 2^e AT' Lxy' + My'^2 + Z^2 + 2^{v-1} T' W^2, & M &= 2BT'L - C^2 L^2, \\ Z &= CLy' + T'z, & 2\mathcal{O} &= T'F'', & P &= T'K. \end{aligned}$$

F'' represents all even integers if and only if $\mathcal{O}$ is universal. If $v = 1$, $M \equiv 1 \pmod{8}$, $T' \equiv 1 \pmod{8}$ we have $\mathcal{O} \equiv y'^2 + Z^2 + W^2 \not\equiv 7 \pmod{8}$. Since $5(y'^2 + Z^2)$ is never congruent to 3, 6 or 7, modulo 8, $\mathcal{O}$ is never congruent to 7, modulo 8, if $M = T \equiv 5 \pmod{8}$. If $M \equiv 1$, $T' \equiv 5 \pmod{8}$ or if $M \equiv 5$, $T' \equiv 1 \pmod{8}$, $\mathcal{O}$ never represents the residue 3 of 8.

It was shown in the second paragraph of the previous page that F'' is never universal when C is even, $T \equiv 2 \pmod{4}$, and $v = 1$. Since, in this case, M is even only when C is even, we have, by the above statement, F'' universal only when M is odd. Hence it is necessary only to consider $M \equiv 1$ or $5 \pmod{8}$, and then, by the preceding paragraph, the only possibility

for universality is $T' \equiv 3 \pmod{4}$. If y' and Z are odd, $W = 0$ then 2, we have $\mathcal{L} \equiv M+1, M+5 \pmod{8}$, and if $W = 0$, $Z = 0$ then 2, $\mathcal{L} \equiv M, M+4 \pmod{8}$. Finally let $Z = 2, W = 1$ $y' = 0$ then 2 and we have $\mathcal{L} \equiv 4+T', T' \pmod{8}$. These residue classes include all integers except multiples of 4.

To proceed by induction from $n \geq 3$ to $n+1$, we let $x = 0$ and $My'^2 + Z^2 + T'W^2 = K + 2^n Q$. Then $M(y'+2^{n-1}\eta)^2 + Z^2 + T'(W + 2^{n-1}\omega)^2 \equiv K \pmod{2^{n+1}}$ has solutions, for, since My' and TW are not both even, the congruence $Q + My'\eta + T'W\omega \equiv 0 \pmod{2}$ has solutions η, ω .

Values of y', Z and W are easily found for which we have $\mathcal{L} \equiv 0, 4 \pmod{8}$ and $\mathcal{L} \equiv 0, 4, 8, 12 \pmod{16}$. Hence we may apply Lemma III, and when $v = 1$, we have $\mathcal{L}$ universal when and only when $M \equiv 1, T' \equiv 3 \pmod{4}$. Hence, under these conditions, F^n represents all even integers.

In case C is even and $v = 2$, M is even and $\mathcal{L}$ is the same type as form (11), page 8, which is treated in Theorem VI, page 15, in which the exponent of 2 in the last term is unity. Hence $\mathcal{L}$ is never universal since $M \not\equiv 4 \pmod{8}$, and hence, in this case F^n is not universal. The case $v > 2$ is excluded by Theorem V, page 10.

The following cases remain: $v = 1, C$ odd, $T \equiv 0 \pmod{8}$; $v = 1, C$ even, $T \equiv 4 \pmod{8}$; $v = 1, C$ even, $T \equiv 8 \pmod{16}$. In the first case we let $T = 8T'$ and we have $F^n \equiv 8Ly^2 + 2q + 2W^2 \pmod{2^e}$, where $q = CLyz + 4T'z^2$. By the argument at the beginning of Section 2, page 3, when y is odd, q runs through all residues of 2^e with z . Hence if we assign odd

values to y and W we have $F'' \equiv P + 2q$, where P is an odd constant. Hence F'' runs through all odd residues of 2^e with q . When $y = 2y'$ we have $F'' \equiv 4BLy'^2 + 4q' + 2W^2 \pmod{2^e}$, where $q' = CLy'z + 2T'z^2$ also runs through a complete residue system of 2^e with z , provided y' is odd. Hence if we assign the odd value π to y and let $W = 0$ then 1, we have $F'' \equiv 4BL\pi^2 + 4q'$, $4BL\pi^2 + 4q' + 2 \pmod{2^{e+1}}$. Evidently as q' runs through all residues of 2^e the two residue classes above will include all even integers. Hence when C is odd, $T \equiv 0 \pmod{8}$, and $v = 1$ F'' is universal.

If $T \equiv 4 \pmod{8}$ or $T \equiv 8 \pmod{16}$ we may set $T = 4T'$. Let $C = 2C'$ and then when $v = 1$ we have $F'' \equiv BLy^2 + 4C'Ly^2 + 4T'z^2 + 2W^2 \pmod{8}$. For C' and T' both odd or both even $F'' \equiv BLy^2 + 2W^2 \not\equiv BL + 4 \pmod{8}$, and, since F'' is odd only when y is odd, it is not universal.

Since F'' is even only when y is even, we let $y = 2y'$, and then $F = 2\lambda$, where $\lambda = 2^eALxy' + 2BLy'^2 + 4C'y'z + 2T'z^2 + W^2$, and hence F represents every even integer $2K$ if and only if λ represents every integer K , or is universal. When T' is odd we have either $T' \equiv BL$ or $BL+2 \pmod{4}$. If C' is even and $T' \equiv BL \pmod{4}$ we have $\lambda \equiv 2BL(y'^2 + z^2) + W^2 \not\equiv W^2 + 6BL \pmod{8}$. Hence, since λ is odd only when W is odd, it is not universal.

If C' is odd and T' is even, we let $T' = 2T''$. Since the only values of T under consideration in this case are

$T \equiv 4 \pmod{8}$ and $T \equiv 8 \pmod{16}$, T'' is odd. Obviously λ is odd only when W is odd then, when y' is odd, we have

$$\lambda \equiv 2BL + 1 \pmod{8}, \text{ and when } y' \text{ is even we have}$$

$\lambda \equiv 4T''z^2 + W^2 \equiv 4T + 1, 1 \pmod{8}$. Hence λ is not universal since it represents only three odd residues of 8.

In the preceeding paragraphs it was shown that when $e \equiv 3$, F'' is universal when C is odd, $T \equiv 0 \pmod{8}$ and $v = 1$; and that $F'' \equiv K \pmod{2^e}$ is solvable when K is odd if C is odd and $T \equiv 2 \pmod{4}$, $v = 1$. It was also shown that F'' represents all even residues of 2^e if C is odd when $M \equiv 1 \pmod{4}$, $T' \equiv 3 \pmod{4}$. Under these conditions on M and T' , it will be seen upon reference to (14), page 20, that we have $T \equiv 6 \pmod{8}$ and $M \equiv 6BL - 1 \equiv 1 \pmod{8}$. The latter congruence yields $BL \equiv 3 \pmod{4}$. In all other cases F'' was shown to be not universal. Hence we have

Theorem IX. Use notations (7) and let $T = HL - 2^\sigma S^2$, $\sigma = v - 2\epsilon - 2$, $\epsilon = u - v$. Except for conditions (6), F is universal when $e \equiv 3$ if and only if either C is odd, $T \equiv 0 \pmod{8}$, $v = 1$; or C odd, $BL \equiv 3 \pmod{4}$, $T \equiv 6 \pmod{8}$, $v = 1$.

5. THE CASE c EVEN, H EVEN, $u \neq v$. In view of Theorem II, page 2, and Theorem III, page 4, we need only to consider cases in which $u > 0$. Let $c = 2C$, $H = 2H'$ and we have

$$(15) \quad F = 2^e Axy + By^2 + 2Cyz + 2H'z^2 + 2^u Szw + 2^v Lw^2.$$

Evidently if F is universal, under these conditions, B

is odd and then F is even only when y is even. Let $e = 1$, and then $F \equiv B \pmod{2}$ is solvable with y odd. If $y = 2y'$ we have $F \equiv 2\varphi$, where

$$\varphi = 2Axy' + 2By'^2 + 2Cy'z^2 + H'z^2 + 2^{u-1}Szw + 2^{v-1}Lw^2.$$

F represents every even integer $2k$ if and only if φ represents every integer k . If H' is odd, evidently $\varphi \equiv 0$, $H' \pmod{2}$ is solvable. If H' is even and $u > 1$, φ is always even and hence is not universal. If $u = 1$, $\varphi \equiv Szw + 2^{v-1}Lw^2 \pmod{2}$. Hence $\varphi \equiv 0, 1 \pmod{2}$ is obviously solvable regardless of v . Hence when $e = 1$, F is universal if and only if B and H' are odd, or if H' is even with $u = 1$.

Let $e \leq 2$, $u \leq 2$ and we have $F \equiv By^2 + 2(Cyz + H'z^2) \pmod{4}$. Then F is odd only when y is odd. If C and H' are both even or both odd, and y is odd $Cyz + H'z^2$ is even regardless of z , and we have only $F \equiv B \pmod{4}$. In order for F to be even, y must be even, and if C is odd and H' is even we have $F \equiv By^2 + 2Cyz \equiv 0 \pmod{4}$, and again F fails to represent an even residue of 4. Hence in either case F fails to represent at least one residue of 4, and consequently our only possibility for universality is C even and H' odd. Let $C = 2C'$ and consider $e > 2$. F is odd only when y is odd and then we have $F \equiv B + 4C'z + 2H'z^2 + 2^uSzw + 2^vLw^2 \pmod{8}$. When z is odd $2^vSzw \equiv 2^vSw \pmod{8}$. Therefore we have $F \equiv B + 4C' + 2H' + 2^uSw + 2^vLw^2 \pmod{8}$ with z odd, and $F \equiv B + 2^uLw^2 \pmod{8}$ with z even. If $u > 2$ these two residues become $B + 4C' + 2H'$ and B , and no other residues are

obtainable. If $u = 2$, $v = 2$ and w is odd, we have, by choice of z , $F \equiv B + 4C' + 2H'$, $B + 4L \pmod{8}$, and when w is even we only obtain one new residue, B , of 8. If $u = 2$, $v > 2$ and w is odd, we have $F \equiv B + 2H'$, $B \pmod{8}$, and when w is even the only additional residue of 8 represented by F is $B + 4C' + 2H'$. In this section we are dealing only with situations in which $u \neq v$, and hence there are no other possibilities. Since in each of the above cases we lack at least one residue of 8, F is not universal when $e > 2$.

When $e = 2$ we first consider $u > 3$, $v > 3$. F is twice an odd only when y is even, z odd. Let $y = 2y'$, then we have $F \equiv 8t + 4By'^2 + 2H'z^2 \pmod{16}$, where $t = Axy' + C'y'z$. Take z odd, then if C' is odd and x is odd, t is even regardless of y' . Hence F represents only the residues $4B + 2H'$, $2H'$ of 16 with z odd. If x is even we obtain no new residues unless t is odd and this can occur only when y' is odd. In that case the only residue of 16 represented by F is $8 + 4B + 2H'$. For C' even, x odd, y' odd and then even we have $F \equiv 8 + 4B + 2H'$, $2H' \pmod{16}$, and for x even, y' as before, $F \equiv 4B + 2H'$, $2H' \pmod{16}$. Hence in either case we have only three of the even residues of 16 which are twice an odd.

In case $u = 3$, $v = 3$ the residues, modulo 16, of $8(Szw + Lw^2)$ must be added to those of the preceding paragraph, but when z is odd the only residue of 16 represented by this form is zero. Hence F is not universal when $e = 2$ if either $u > 3$, $v > 3$ or $u = 3$, $v = 3$.

If $e = 2$, $u = 3$, $v > 3$ and y is odd, we have, since C is even, $F \equiv B, B+2H' \pmod{4}$, by choice of z . If $y = 2y'$, y' odd, $F \equiv 4 + 2H', 4 \pmod{8}$ is solvable. Next let $y = 4y'$ and we have $F \equiv 2H'z^2 + 8Szw \equiv 0, 2H', 8H', 2H' + 8 \pmod{16}$ by suitable choice of the variables z, w . All integers are included in the above residue classes.

If $u = 2$, $v \leq 2$, F represents the two odd residues $B, B+2H'$ of 4 if we take y odd, $w = 0$, $z = 0$ then 1. If $y = 2y'$, y' odd, $z = w = 0$, then $z = 1, w = 0$ we have $F \equiv 4B, 4B+2H' \pmod{8}$. Next let $y = 4y'$, y' odd, $w = 0$ and let z take successively the values, 0, 1, 2, then with y unchanged let $z = 1, w = 2$ and we obtain the congruences $F \equiv 0, 2H', 8H', 2H' + 8S \pmod{16}$. The above residue classes of 8 and 16 include all even integers. Thus when $u \leq 2$, $v \leq 2$, and we except conditions (6), F is universal only when $e = 2$, $C \equiv 0 \pmod{4}$, $H \equiv 2 \pmod{4}$ and either $u = 3$, $v > 3$ or $u = 2$, $v \leq 2$.

The general cases $u = 0$, $u > v$ have been disposed of in earlier sections and we therefore have only the cases $u=v=1$ and $u = 1$, $v > 1$ to consider. In the latter case we have $F \equiv By^2 + 2Cyz + 2H'z^2 + 2t \pmod{2^e}$ where $t = Szw + 2^{v-1}Lw^2$. It was shown in the first paragraph of section 2 that when z is odd, t runs through all the residues of 2^e with w . Hence when y and z are fixed odds, $F \equiv K + 2t \equiv G \pmod{2^e}$ is always solvable for w when G is odd. If $y = 2y'$ with y' and z odd we have $F \equiv P + 2t \equiv G \pmod{2^{e+1}}$ always solvable for w when G is even. Hence by Lemma II F is universal if we except conditions (6).

If $u = 1$, $v = 1$ we first consider the case H' even, $e \leq 2$. F represents the two residues $B+2$ and B of 4 with y odd, $z = 0$, $w = 1$ then $z = w = 0$. If C is even we have $F \equiv B \pmod{4}$ with y odd, $z = 1$, $w = 0$, but if C is odd we have $F \equiv B \pmod{4}$ with $z = 0$, $w = 0$. We next let $y = 2y'$ with y' odd and take $z = 0$, $w = 1$ then $z = w = 1$ and we have $F \equiv 2L, 0 \pmod{4}$. These residues constitute a complete set, modulo 4.

To proceed by induction let $x = 0$ and $F = By^2 + 2Cyz + 2H'z^2 + 2Szw + 2Lw^2 = K + 2^nQ$. Then $F(0, y, z + 2^{n-1}\xi, w + 2^{n-1}\omega) = K + 2^n(Cy\xi + Sw\xi + Sz\omega + Q) + \mu$, where μ is a form in which each coefficient contains, as a factor, 2 with an exponent $n+1$ or $2n-1$. $Cy\xi + Sw\xi + Sz\omega + Q \equiv 0 \pmod{2}$ is solvable since Cy, Sw, Sz are not all even. Then $F \equiv K \pmod{2^n}$ is always solvable if $n \leq 2$, with K arbitrary.

If H' is odd, $e = 2$, we have $F \equiv B, B+2L \pmod{4}$ with y odd, $z = 0$, $w = 0$ then 1. Let $y = 2y'$, y' odd, set $z = 0$, $w = 0$ then 1. Next with y unchanged, let $z = 2$, $w = 1$ and we have $F \equiv 4B, 4B+2L, 2L \pmod{8}$. If $y \equiv 4y'$, $z = 0$, $w = 0$ then 2 we have $F \equiv 0, 8 \pmod{16}$. These residue classes include all integers.

If H' is odd and $e \leq 3$ and C is even, F is congruent to B , modulo 4, only when z and w are both even, and then F is never congruent to $B+4$, modulo 8. If C is odd $F \equiv B+2 \pmod{4}$ is solvable only with y odd, z even and w odd, and under these conditions, F is congruent only to $B+2L$, modulo 8, and hence represents only one residue of 8 which is of the

form B plus twice an odd. Since F is odd only when y is odd, all odd residues represented by F must be of the form $B+2p$. Hence, in either case, F is not universal.

This completes proof of

Theorem X. If c is even and H is even, $1 \leq u \leq v$ and cases (6) are excluded, F is universal if and only if B is odd and the following conditions hold: $e = 1$, H' odd; $e = 1$, H' even, $u = 1$; $e = 2$, $c \equiv 0 \pmod{4}$, $H \equiv 2 \pmod{4}$ and either $u = 3$, $v > 3$, or $u = 2$, $v \leq 2$; $e \leq 2$, $u = 1$, $v > 1$; $e = 2$, $u = 1$, $v = 1$; $e \leq 3$, $u = 1$, $v = 1$, $H \equiv 0 \pmod{4}$; where $H = 2H'$.

6. THE CASE c EVEN, H ODD, $u \leq v$. If $e = 1$, we have $F \equiv B, B + H \pmod{2}$ solvable with y odd, and hence F is universal.

Let $c = 2C$. Then $F \equiv G \pmod{2^n}$ is solvable when G and n are arbitrary if and only if $HF \equiv HG \pmod{2^n}$ is solvable, where $HF = 2^e AHxy + \gamma y^2 + Z^2 + 2^u HSzw + 2^v LW^2$, $Z = Hz + Cy$, $\gamma = HB - C^2$. If $e \leq 2$, $u \leq 2$, $v \leq 2$, HF represents only the residues $0, 1, \gamma$, and $\gamma+1$ of 4 , and these differ only when $\gamma \equiv 2 \pmod{4}$. Therefore when C is odd, F is universal only when $BH \equiv 3 \pmod{4}$ or $H \equiv 3B \pmod{4}$, and when C is even we have $B \equiv 2 \pmod{4}$. Under these conditions HF is odd only when Z is odd, and if $e \leq 3$, or $u > 2$, $v > 2$, HF is congruent, modulo 8 , only to the odd residues 1 and $\gamma+1$, and hence is not universal. If $e=2$, $u > 2$, $v > 2$, we have $HF \equiv 1, \gamma+1, \gamma+5 \pmod{8}$ when Z is odd and hence is not universal.

By the above paragraph, B is odd when C is odd if $u \leq 2$, $v \leq 2$, and it is obvious that F is odd only when either y is

even and z is odd or y is odd and z is even. If $e \leq 2$ and $u = v = 2$ we first take y odd and let $z = 2z'$ and we have $F \equiv 4(2^{e-2}Axy + Lw^2) + 4(Cz' + Hz'^2) + B \pmod{8}$. Evidently, under these conditions F is at most congruent to B and $B+4$, modulo 8. We next let $y = 2y'$ and take z odd and we have $F \equiv 4(By'^2 + Cy'z + Szw + Lw^2) + H \pmod{8}$. Since for odd values of z the expression in the parenthesis is always even, we only have $F \equiv H \pmod{8}$. Hence F represents only three of the four odd residues of 8 and hence is not universal,

In case $e \leq 3$, $u = 2$, $v > 2$, and C is odd, take y odd and let $z = 2z'$, and then $F \equiv B + 4(Cz' + Hz'^2) + 2^v Lw^2 \equiv B + 2^v Lw^2 \pmod{8}$. When $y = 2y'$ and z is odd, we have $F \equiv 4(By'^2 + Cy') + 4Sw + 2^v Lw^2 + H \equiv 4Sw + 2^v Lw^2 + H \pmod{8}$. When $v > 2$ and w is taken even and then odd, the above residues become B , H , $4S + H$. Hence F fails to represent one of the odd residues of 8 and is not universal.

If $e = 2$, $u = 2$, $v \leq 3$, F is congruent, modulo 4, to B , with y odd and z even, and to $B + H + 2C$, modulo 4, with z odd. Since $H \equiv 3B \pmod{4}$, when C is odd, the latter residue is congruent to $2C$, modulo 4. Let $y = 2y'$ and first take z odd, w even then odd, and next let $w = 0$, and take $z = 0$ then 2 and we have $F \equiv H, H + 4S, 4B, 0 \pmod{8}$. Since $H \equiv 3B \pmod{4}$, the above residue classes include all integers. Hence when $u \leq 2$, $v \leq 2$ and C is odd, F is universal only when $e = 2$, $u = 2$, $v \leq 3$.

When C is even, $u = 2$, $v > 2$, the only possibility for universality is $B \equiv 2 \pmod{4}$, as was shown in the first

paragraph of this section. Hence $B = 2B'$ where B' is odd.

Let $C = 2C'$ and we have

$$F = 2^e Axy + 2B'y^2 + 4C'yz + Hz^2 + 4Szw + 2^v Lw^2.$$

Evidently F is odd only when z is odd. If $e = 2$, $F \equiv 2B' + H \pmod{4}$ has solutions with y odd, and for $y = 2y'$, y' odd, $z = 1$, $w = 1$ then 0 , $F \equiv 4S + H, H \pmod{8}$. These residue classes include all odd integers. If $e = 3$, $F \equiv H + 2B'$, $H + 2B' + 4 \pmod{8}$ has solutions with y odd. If C' is odd and $y = 2y'$, y' odd, $z = 1$, and w takes successively the values $0, 1, 2$, and 3 , we have $F \equiv H, H + 4S + 2^v L, H + 8S$, and $H + 12S + 9(2^v L) \pmod{16}$. These residue classes include all odd integers if $v \leq 2$. If C' is even 8 must be subtracted from each of the above residues of 16 , the only effect of which would be a permutation.

Since F is even only when z is even, we let $z = 2z'$, then $F = 2\phi$, where

$$\phi = 2^{e-1} Axy + B'y^2 + 4C'yz' + 2Hz'^2 + 2^2 Sz'w + 2^{v-1} Lw^2.$$

Hence F represents all even integers if and only if ϕ is universal. Evidently the g. c. d. of the coefficients of the last three terms of ϕ is twice an odd. B' is odd and hence each of its factors is odd, and the coefficient of yz is even. First consider $v > 2$. It was shown by Prof. Dickson¹ that such a form is universal when $e' = e - 1 = 1$. Since $u = 2$, $c' = 4C' \equiv 0 \pmod{4}$, and the coefficient of z'^2 is congruent to 2 , modulo 4 , we have, by Theorem X, when $e' > 1$ that a form of the type of ϕ is universal only when $e-1 = 2$.

1. Transactions of the A.M.S., vol. 31, no. 1, page 177, Theorem 9.

Hence when $e > 1$, ϕ is universal only when $e = 2$ or 3 . Since it was shown in the previous paragraph that $F \equiv G \pmod{2^e}$ is solvable under these conditions when G is odd, and since the conditions of Lemma II are satisfied, we have F universal only when $e = 2$ or 3 .

If $u = 2$, $v = 2$ and $e \leq 3$ and z is odd, F is congruent only to $2B' + 4C' + H$, $H \pmod{8}$. Hence, since F is odd only when z is odd it is not universal. If $e = 2$ the only possibility of F representing additional odd residues of 8 would occur when x and y are odd. This would contribute only one additional residue, $4 + 2B' + 4C' + H$, and hence again F is not universal.

If $u = 1$, $v > 1$ $F = 2^e Axy + By^2 + 2Cyz + Hz^2 + 2t$, where $t = Szw + 2^{v-1}Lw^2$. If B is odd the argument made in the last paragraph of page 25 is valid here, excepting that F represents even integers when y is odd and odd integers when y is even. If B is even we may show that F represents all odd integers by the same argument, but since F is even only when z is even, we must use another method to determine the conditions under which F represents all even integers. Let $z = 2z'$, $B = 2B'$ and we have $F = 2\phi$, where
$$\phi = 2^{e-1}Axy + B'y^2 + 2Cyz' + 2Hz'^2 + 2Sz'w + 2^{v-1}Lw^2.$$
 ϕ is never odd if B' is even, and in this case F would never represent evens which are twice an odd. Hence B' is odd. Prof. Dickson¹ has shown that a form of the type of

1. Transactions of the A. M. S., vol. 31, no. 1, page 177, Theorem 9.

ϕ is universal when $e' = e - 1 = 1$. If $e' > 1$, ϕ is universal by Theorem X, page 27, if $v' = v - 1 > 1$; and if $v' = 1$, ϕ is universal if $e' = 2$, but if $e' > 2$, ϕ is universal only when $2H \equiv 0 \pmod{4}$. The last condition is not satisfied since H is odd.

Theorem XI. Let $c = 2C$ and let H be odd. When $e = 1$ F is universal if and only if cases (6) are excluded. If $e \geq 2$, $u \geq 1$, $v > 1$, but $u < v$, F is universal if and only if we exclude cases (6) and the following conditions hold: $e = 2$, C odd, $H \equiv 3B \pmod{4}$, $u = 2$, $v \leq 3$; $e = 2$ or 3 , C even, $B \equiv 2 \pmod{4}$, $u = 2$, $v > 2$; B odd, $u = 1$, $v > 1$; $B \equiv 2 \pmod{4}$, $u = 1$ and either $e = 2$, $v > 1$ or $e = 3$, $v \leq 2$ or $e > 3$, $v > 2$.

In case $u = 1$, $v = 1$ we have

$F = 2^e Axy + By^2 + 2Cyz + Hz^2 + 2Ssw + 2Lw^2$. If B is even and C is odd we first consider $B \equiv 2 \pmod{4}$. F is odd only when z is odd, and if $z = \nu$, ν odd, we have $F = H\nu^2 + q$ where $q = 2^e Axy + By^2 + 2Cy\nu + 2(S\nu w + Lw^2)$. When $e \geq 2$, we have $q \equiv 0 \pmod{4}$ regardless of the values of y and w . Hence F never represents an integer of the type $H + 2k$, when k is odd, and is therefore not universal. If $B \equiv 0 \pmod{4}$ and C is odd, we have the congruence $F \equiv 2C + H, 2L, 0 \pmod{4}$ solvable with y odd. If $y = 2y'$ the congruence $F \equiv 4C + H + 2(S + L), 4C + H + 2(3S + L) \pmod{8}$ are solvable with y' odd. The residue classes $2L$ and 0 of 4 include all even integers, and the class $2C+H$ of 4 includes all odds of the form H plus twice an odd, while the classes $H + 4C + 2(S + L), H + 4C + 2(3S + L)$ of 8 include all in-

clude all integers of the form H plus multiples of 4. The conditions of Lemma II are satisfied and hence when $e = 2$ F is universal.

When B and C are both even we first consider the case $B \equiv 0 \pmod{4}$. If z is odd we have $F \equiv H + 2(Sw + Lw^2) \equiv H \pmod{4}$, and, since F is odd only when z is odd, it represents only one odd residue of 4, and hence is not universal. If $B \equiv 2 \pmod{4}$ the congruences $F \equiv B+H$, $B+2L$, $B \pmod{4}$ are solvable with y odd, and if $y = 2y'$, y' odd, $z = 1$, $w=1$ then 3 we have $F \equiv H + 2S + 2L$, $H + 6S + 2L \pmod{8}$. All integers are included in these residue classes and the conditions of Lemma II are satisfied, and hence F is universal.

When B is odd and $e = 2$, we take y odd and $z=0$ and we have $F \equiv B+2L$, $B \pmod{4}$ by choice of w . These residue classes include all odd integers. If $z = 1$, $w = 0$ we have $F \equiv B + H + 2C \pmod{4}$. If $B + H + 2C \equiv 2 \pmod{4}$ let $y = 2y'$ take y' odd, $w = 0$, and then by choice of z we have $F \equiv 0$, $4B \pmod{8}$. If $B + H + 2C \equiv 0 \pmod{4}$ we take y as before, let $z = 0$ and $w = 1$, and we have $F \equiv 4B+2L \pmod{8}$, and if $y = 4y'$ with y' odd, let $w = 1$, and by choice of z we have $F \equiv 2L$, $8S + 2L \pmod{16}$. The conditions of Lemma II are satisfied and hence F is universal.

Hence we have

Theorem XII. If $c = 2C$, $u = v = 1$, H is odd and $e = 2$, F is universal when B is odd, but if B is even, F is universal only in the following cases: C odd, $B \equiv 0 \pmod{4}$; C even, $B \equiv 2 \pmod{4}$. In all cases universality is qualified by ex-

cluding conditions (6).

If $e \equiv 3$ we first consider B even and C odd. It was shown on the preceeding page that we need only to consider $B \equiv 0 \pmod{4}$. Let $B = 4B'$ and then we have

$$F = 2^e Axy + 4B'y^2 + 2Cyz + Hz^2 + 2t, \quad t = Szw + 2Lw^2.$$

Let $y = \pi$, where π is an odd prime, and since F is odd only when z is odd, we let $z = \nu$, ν odd. Then by the argument on page 4, t will run through all the even residues of 2^e with w . Hence $F \equiv 4B' + 2C\pi\nu + H$, $4B' + 2C\pi\nu + H + 4 \pmod{8}$ by choice of t . These residue classes include all odd integers which are of the type H plus twice an odd. Now let $y = 2y'$, $y' = \pi$, $z = 1$, and let t run through an even residue system of 16 and we have $F \equiv H, H+4, H+8, H+12 \pmod{16}$. These residue classes include all integers which are of the type H plus multiples of 4.

To proceed by induction from $n \equiv 3$ to $n+1$, we let $x = 0$ and $F = 2^n Q + k$. Then $F(x, y, z, w + 2^{n-1}\omega) \equiv k \pmod{2^{n+1}}$ is solvable, for, since Sz is odd, $Q + Sz\omega \equiv 0 \pmod{2}$ is solvable for ω . Hence $F \equiv k \pmod{2^e}$ is always solvable when k is odd.

If $z = 2z'$ we have $F = 2\phi$, where

$\phi = 2^{e-1}Axy + 2B'y^2 + 2Cyz' + 2Hz'^2 + 2Sz'w + Lw^2$. If the notation of Theorem II is expressed in terms of the coefficients of ϕ , we have $gb' = 2B'$, $b' = \frac{1}{2}b$, $C=C$, $gdj = 2gds=2S$, $gdr' = L$, $r' = \frac{1}{2}r$, $gdh' = 2H$, $h' = 2h$, $\lambda=2HL - S^2$, $e' = e-1$, $M' = 2BL\lambda - L^2C^2$. Since L is odd, d and r' are odd; S is odd and hence λ is odd; j is even and $e' = 2$. Hence the hy-

pothesis of the theorem is satisfied, and M' is odd. Hence φ is universal if and only if we exclude cases (6) and $\lambda \equiv M' \equiv 1 \pmod{4}$ with $e' \equiv 3$. Otherwise F represents all even integers and we have

Theorem XIII. If $c = 2C$, $u = v = 1$, H is odd, $e \equiv 3$, B is even and C is odd, F is never universal when $B \equiv 2 \pmod{4}$. Let $B = 4B'$, $\lambda = 2HL - S^2$, $M' = 2B'L\lambda - L^2C^2$, then F is universal if and only if we exclude conditions (6) and the case $\lambda \equiv M' \equiv 1 \pmod{4}$, $e \equiv 4$.

When B and C are both even it was shown in the second paragraph of page 32 that F is never universal if $B \equiv 0 \pmod{4}$. Hence $B = 2B'$ where B' is odd, and if $C = 2C'$ we have $F = 2^e Axy + 2B'y^2 + 4C'yz + Hz^2 + 2t$, where t has the same value as above. It has been shown that t runs through all even residues of 2^e with w when z is odd. Hence when y is odd, $F \equiv H + 2B'$, $H + 2B' + 4 \pmod{8}$ has solutions, and if $y = 2y'$, y' odd, the congruences $F \equiv H, H+4, H+8, H+12 \pmod{16}$ are solvable, with z odd.

The induction argument given on the previous page will hold here since Sz is odd.

Since F is even only when z is even, we let $z = 2z'$ and then $F = 2\psi$, where

$$\psi = 2^{e-1}Axy + B'y^2 + 4C'yz' + 2Hz'^2 + 2Sz'w + Lw^2.$$

In this case, if the notation of Theorem II is expressed in terms of the coefficients of ψ we have $gb' = B'$, $b' = \frac{1}{2}b$, $C = 2C'$, $gdh' = 2H$, $h' = 2h$, $gdj = 2gds = 2S$, $gdr' = L$, $r' = \frac{1}{2}r$, $\lambda = 2HL - S^2$, $M' = \lambda B'L - 4L^2C'^2$. Since L is odd, g and d

are odd, j is even, c is even and λ is odd. Hence M' is odd. The hypothesis of the theorem is satisfied and we have ψ universal if and only if we exclude cases (6) and the case

$\lambda \equiv M' \equiv 1 \pmod{4}$, $e' = e - 1 \equiv 3$. Hence F represents all even integers if and only if the above cases are excluded.

This completes the proof of

Theorem XIV. If $c = 2C$, $u = v = 1$, H is odd, $e \equiv 3$, B and C are even, F is never universal if $B \equiv 0 \pmod{4}$. If $B = 2B'$, B' odd, F is universal if and only if we exclude cases (6) and the case $\lambda \equiv M' \equiv 1 \pmod{4}$, $e \equiv 4$, where $\lambda = 2HL - S^2$, $M' = \lambda B'L - 4L^2C'^2$, $C = 2C'$.

If B and C are both odd and $e \equiv 3$, $F \equiv G \pmod{2^e}$ is solvable if and only if $BF \equiv BG \pmod{2^e}$ has solutions, where $BF = 2^e ABxy + B^2y^2 + 2BCyz + BH^2z^2 + 2BSzw + 2BLw^2$. BF is odd only when either y is odd and z is even or when y is even and z is odd. When z is even and w is even we have $BF \equiv 1 \pmod{8}$. If w is odd and z is twice an odd then twice an even we have $BF \equiv 5+2BL, 1+2BL \pmod{8}$. Obviously these residues are congruent, aside from order to 1, 3, 7 of 8. When $y = 2y'$ and z is odd $BF \equiv 4(y'^2 + BCy'z) + BH^2z^2 + 2t \pmod{8}$ where $t = Bszw + BLw^2$. Obviously when z is odd, t is even and hence in this case the only residues of 8 represented by BF are $BH, BH+4$. Hence BF represents all odd residues of 8 only when $BH \equiv 1 \pmod{4}$ holds.

When y and z are both odd and we let w take the values 1, then 3 we have, aside from order, $BF \equiv 0, 4 \pmod{8}$. Hence BF represents all odd residues of 8 and all residues which

are multiples of 4.

To proceed by induction from $n=3$ to $n+1$ we let $x = 0$ and $BF = K + 2^n Q$. Then $F(0, y + 2^{n-1} \eta, z + 2^{n-1} \xi, w) = F(0, y, z, w) + 2^n (Q + B^2 y \eta + BCy \xi + BCz \eta + BH z \xi + BSz \xi) + \tau$, where the coefficients of τ are divisible by 2^{2n-2} or by 2^{2n-1} . If y and z are not both even $B^2 y \eta + BCy \xi + BCz \eta + BH z \xi + BSz \xi \equiv 0 \pmod{2}$ is solvable and hence by choice of η, ξ we have $BF \equiv K \pmod{2^{n+1}}$. Hence the induction is complete.

It is now necessary to prove that, under the above conditions, BF represents all even integers which are multiples of 2. We let $y = 2y'$, $z = 2z'$ and then we have $BF = 2\rho$, where $\rho = 2^e ABxy' + 2B^2 y'^2 + 4BCy'z' + 2BH z'^2 + 2BSz'w + BLw^2$. Hence BF represents all integers which are twice an odd if and only if ρ represents all odds. If $e = 3$, w is odd, $y = 0$, $z = 0$ then 2, then with w and z as before let $y = 1$ and we have $\rho \equiv BL, BL+4, BL+2, BL+4+2 \pmod{8}$. This is a complete odd residue set of 8.

To proceed by induction from n to $n+1$ we let $x=0$ and $\rho = K + 2^n Q$. Then $F(0, x, y, 2^{n-1} \omega + w) = K + 2^n (Q + BSz' \omega + BLw \omega) + 2^{2n-2} BL \omega^2$. When z is even and w is odd, $Q + BSz' \omega + BLw \omega \equiv 0 \pmod{2}$ has solutions. Hence by choice of ω we have $\rho \equiv K \pmod{2^{n+1}}$. This completes the proof of

Theorem XV. Employ notation (7). If $c = 2C$ and $C, H,$ and B are odd, F is universal when $e \geq 3$ if and only if $BH \equiv 1 \pmod{4}$ holds, and the cases (6) are excluded.

In case B is odd and C is even we let $C = 2C'$, and then we have $BF \equiv y^2 + 4BC'yz + BH^2z^2 + 2BSzw + 2BLw^2 \pmod{8}$. BF is odd only when one of y, z is odd and the other even. We first let $z = 2z'$ and the congruence $BF \equiv y^2 + 4(BH^2z'^2 + BSz'w) + 2BLw^2 \pmod{8}$ holds. If y is odd, obviously BF only represents the residues $1+2BL, 1+4BH, 1 \pmod{8}$. If $y = 2y'$ and z is odd, we have $BF \equiv 4y'^2 + BH^2z^2 + 2t \equiv BH, BH+4 \pmod{8}$, since $t = BSzw + BLw^2$ is always even when z is odd. Hence BF represents all four odd residues of 8 only when $BH \equiv 3 \pmod{4}$. If y and z are both odd we have $BF \equiv 1+4BC' + BH + 2t \equiv BH+1, BH+5 \pmod{8}$ by choice of t. If y and z are both even and w is odd we have $BF \equiv 4 + 2BL, 2BL \pmod{8}$. Since $BH \equiv 3 \pmod{8}$ the two residues $BH+1$ and $BH+5$ are multiples of 4, while each of the two residues in the previous sentence is the double of an odd. Hence we have a complete set of residues of 8 if and only if the congruence $BH \equiv 3 \pmod{8}$ holds.

To proceed by induction from $n \geq 3$ to $n+1$ we let $x = 0$ and $F = K + 2^n Q$, where K is arbitrary. Then $F(0, y+2^{n-1}\eta, z+2^{n-1}\xi, w+2^{n-1}\omega) \equiv 0 \pmod{2^{n+1}}$ has solutions ξ, η, ω if $Q + y\eta + BSz\omega + BH^2\xi + BS\omega\xi \equiv 0 \pmod{2}$ has solutions. The latter congruence is solvable since y, BS, BH^2 , BS ω are not all even. Hence the induction is complete and we have

Theorem XVI. If $c = 2C$, H is odd and $u = v = 1$ and $e \geq 3$, F is universal when B is odd if and only if C is even and $BH \equiv 3 \pmod{4}$.

7. THE CASE c ODD. In this case it is necessary to investigate all possibilities on u and v except the case $u = 0$, which was treated in Section 2. We first consider H odd. If $e \geq 1$, $u > 0$, $v > 1$ and B is odd, F is even only when y and z are both even and then we have $F \equiv 0 \pmod{4}$. Hence F is not universal since it represents only multiples of 4. If $e = 1$, $v = 0$, let $z = 0$ and we have $F \equiv By^2 + Lw^2 \equiv B + L, B \pmod{2}$ with y odd, $w = 1$ then 0. This is a complete residue set of 2 and hence F is universal. If $v = 1$, y is odd, $z = w = 0$, we have $F \equiv B \pmod{2}$. If $y = 2y'$, y' odd, $z = 0$, $w = 0$ then 1 we have $F \equiv 0, 2 \pmod{4}$, and hence by Lemma II F is universal. If B is even, $u > 0$, let $w = 0$ and then we have $F \equiv 0 \pmod{2}$ with y and z odd. Let z be odd and let $y = 2y'$ and then $y = 4y'$, with y' odd in each case, and we have $F \equiv H+2c \pmod{4}$ and $F \equiv H+4c \pmod{8}$. Next let $y = 8y'$, with y' odd, and take z equal 1 then 3, and we have $F \equiv H + 8c, H \pmod{16}$. These residue classes include all integers and the conditions of Lemma II are satisfied. Hence we have

Theorem XVII. We employ notation 7. When c and H are odd and $e = 1$, F is universal when $u > 0$ if B is even, but when B is odd F is universal if and only if $v = 0$ or $v = 1$.

If $e > 1$ we first consider B odd and the case $u = v = 1$. Let $t = cyz + Hz^2$. It was shown in the second paragraph of page 4 that a form of the type of t will run through a complete even residue system of 2^e as z runs through either a complete odd or even residue system of the same modulus,

provided y is odd. Let $w = 0$ and we have $f = 2^e Axy + By^2 + t$, then when y is odd, F runs through a complete odd residue system of 2^e as t runs through the even residues of the same modulus. Hence $F \equiv G \pmod{2^e}$ is always solvable when G is odd.

It was shown above that F is never even unless y and z are both even. If $y = 2y'$ and $z = 2z'$ we have $F = 2\phi$, where $\phi = 2^e Axy' + 2By'^2 + 2cy'z' + 2H'z'^2 + 2Sz'w + Lw^2$. F will represent all even integers $2K$ if and only if ϕ represents every integer K , that is, if ϕ is universal. It will be seen upon referring to (7) that $2B = 2bg = gb'$, $b' = 2b$; $c' = 2c$; $2H = 2gdh = gdh'$, $h' = 2h$; $2S = gdj$, then if $j = 2s$, $S = gds$; $2L = gdr$. Since S is odd, g, d and z are odd and hence r must be even. Let $r' = \frac{1}{2}r$, and hence $L = gdr'$. Since L is odd, r' is odd. Using the accented notation in Theorem II we have $c' = 2c$, $\lambda = g^3 d^3 (r' h' - s^2) = 2HL - S^2$, $M = \lambda g^2 db' r' - g^2 d^2 r^2 c^2 = \lambda 4EL - L^2 c^2$. Obviously λ is odd and, since d and l are odd, the hypothesis of Theorem II is satisfied. Hence, since M is odd, we have ϕ universal if and only if we exclude cases (6) and $\lambda \equiv M \equiv 1 \pmod{4}$, $e = 4$.

If $u > 0$, $v = 0$ we have $F = 2^e Axy + p + Hz^2 + 2^u Szw + Lw^2$, where $p = By^2 + cyz$. Obviously p is the same type of form as r which was described on the preceding page. Let $e = 2$. If we assign an odd prime to y and let $w = 0$ then $w = 1$, we have, by choice of p , $F \equiv H, H+2, H+L, H+L+2 \pmod{4}$. This constitutes a complete residue set of 4.

If $e \geq 3$, let $w=0$ and let z run through an odd res-

idue system of 8, and then we have $F \equiv H, H+2, H+4, H+6 \pmod{8}$, where u and v have the same value as in the above paragraph. If $u > 2$, $v = 0$, we let $w = 1$ and take z as before and we have $F \equiv H+L, H+L+2, H+L+4, H+L+6 \pmod{8}$, but if $u = 2$, $v = 0$, we must add 4 to each of these residues, and this would only cause a permutation. If $u = 1$ and $v = 0$ and z is odd, odd values of w may be determined so that $2Szw \equiv 2, 6 \pmod{8}$ holds. Determine an odd value of z so that the congruence $p \equiv 2 \pmod{8}$ holds, and then by choice of w we have $F \equiv H+L, H+L+4 \pmod{8}$. Next determine z so that we have $p \equiv 4 \pmod{8}$, and again by choice of w we have $F \equiv H+L+6, H+L+2 \pmod{8}$.

To obtain an induction on n we let $x = 0$ and $F = G+2^n Q$. Then $F(0, y, z+2^n \xi, w) = F(0, y, z, w) + 2^n(cy\xi + Q) + \delta$, where the coefficients of δ contain powers of 2 greater than or equal to $n+1$ for values of $n \geq 2$. Since cy is odd we have $cy\xi + Q \equiv 0 \pmod{2}$ solvable, and hence $F(0, y, z+2^n \xi, w)$ is congruent to G , modulo $n+1$, by choice of ξ , and the induction is complete.

This completes proof of
Theorem XVIII. Employ notation 7. If B, c and H are odd, $u \leq 1$, and $e \leq 2$, k is universal if and only if we exclude cases (6) and the following: $u \leq 1, v \leq 2; \lambda \equiv M \equiv 1 \pmod{4}, e \leq 4$, where $\lambda = 2HL - S^2, M = 4BL\lambda - L^2c^2$.

We now let $B = 2B'$ and consider $e \leq 2$. If $w = 0$ and $y = \pi$, where π is odd, and we let $z = 1$ then 3, we have

$F \equiv 2B' + c\pi + H$, $2B' + 3c\pi + H \pmod{4}$, then with w unchanged, we let $z = 1$, $y = 0$, then $y = 2$ and we have $F \equiv H$, $H+2c \pmod{4}$. Hence we have a complete residue set of 4.

To proceed by induction from $n \geq 2$ to $n+1$, we let $x = 0$, and $F = G + 2^n Q$. Then $F(0, y+2^n \gamma, z, w) = F(0, y, z, w) + 2^n(Q + cz\gamma) + \beta$, where the coefficients of β contain powers of 2 which are greater than or equal to $n+1$, for values of $n \geq 2$. Since cz is odd, $Q + cz\gamma \equiv 0 \pmod{2}$ is solvable, and hence $F(0, y+2^n \gamma, z, w) \equiv G \pmod{2^{n+1}}$ has a solution. Hence the induction is complete and we have

Theorem XIX. We employ notations (7). If B is even, and H is odd, and $e \geq 2$, F is universal if and only if we exclude cases (6).

We finally consider H even. Let $H = 2H'$, $w = 0$, and we have $F = 2^e Axy + By^2 + T$, where $T = cyz + 2H'z^2$. By the argument in the first paragraph of Section 2 it may be shown that when y is odd, T runs through a complete residue set of the modulus 2^e with z . Hence if we assign to y the odd prime value π , we have $F \equiv G \pmod{2^e}$ always solvable. Hence we have

Theorem XX. We employ notation (7). F is universal when c is odd and H is even if and only if we exclude cases (6).

APPENDIX.

The symbol D(11112247) is used to represent the form $x_1'' + x_2'' + x_3'' + x_4'' + 2x_5'' + 2x_6'' + ax_7'' + 7x_8''$. Professor Dickson has shown that this form represents all integers less than or equal to 1300, and it is the purpose of this paper to show that D represents all integers less than or equal to 4184. In obtaining the representations exhibited below, use was made of the Waring Table of Fourth Powers made by Derrick Norman Lehmer. We shall use the following notation which was employed by Mr. Lehmer: $p = 10$, $q = 11$, $w = 12$, $E = 13$, $r = 14$, $t = 15$, $y = 16$, $u = 17$, $i = 18$, $o = 19$. We also use $a = 2^4$, $b = 3^4$, $c = 4^4$, $d = 5^4$, $e = 6^4$, $f = 7^4$, $h = 8^4$, and we employ the following symbols for certain combinations which occur frequently:

A(x) for $x, 1+x, 2+x, ---a+x, 1+a+x, ---2a+x, ---14+4a+x$.

B(x) for $2x, 1+2x, ---a+2x, ---14+3a+2x$.

C(x,y) for $x+y, 1+x+y, ---a+x+y, ---14+3a+x+y$.

G(x,y) for $4x+y, ---14+4x+y$.

H(x) for $3x, ---14+2a+3x$.

I(x,y,z) for $2x+2y+z, ---14+2x+2y+z$.

J(x,y) for $2x+4y, ---13+2x+4y$.

K(x,y) for $x+3y, ---14+a+x+3y$.

L(x,y,z) for $x+y+4z, ---13+x+y+4z$.

M(x,y,z,w) for $x+y+z+2w, ---14+x+y+z+2w$.

N(x) for $4x, ---15+4x, a+4x, ---14+a+4x$.

O(x,y,z) for $x+y+z, ---a+x+y+z$.

$P(x,y)$ for $2x+2y, \dots 14+a+2x+2y$.

$Q(x,y)$ for $2x+y, \dots 14+2a+2x+y$.

$R(x,y,z)$ for $2x+y+z, \dots 14+a+2x+y+z$.

In the following table the series of integers 1,2,...78 is shown to have the representation $A(1)$. The representation 08013 is exhibited for the integer 2259, and this means that $2259 = 0.1 + 8a + 0.b + c + 3d$. The statement that the series of integers 1686,...1682 has the representation 0472---6472 means that each of the integers 1686, 1687,...1692 is represented respectively by 0472, 1472,...6472. It frequently happens that the first part of a series of integers will be represented by one of the combinations described above and the latter part by another of the combinations, while some of the integers of the series have representations in both combinations. Such a situation is illustrated by the statement that the series of integers 243-334 has the representation $H(b) - A(c)$.

<u>Integer</u>	<u>Representation</u>	<u>Integer</u>	<u>Representation</u>
1-78	$A(1)$	240-2	$0t-2t$
79	$t4$	243-334	$H(b)-A(c)$
80	05	335	$q04$
81-159	$A(b)$	336	$w04$
160	$0p$	337-399	$C(b,c)$
161	$1p$	400-404	$0901-4901$
162-224	$B(p)$	405-417	$005-w05$
225	091	418-464	$Q(b,c)$
226-239	$042-E042$	465-474	$0811-9811$

<u>Integer</u>	<u>Representation</u>	<u>Integer</u>	<u>Representation</u>
475-479	9321-E321	985-992	0091-7091
480	0r01	993	07011
481-485	0911-4911	994-1008	M(b,c,d,a)
486-498	006-w06	1009-1018	08011-98011
499-574	K(c,b)-B(c)	1019-1023	2291-6291
575	807	1024-1054	N(c)
576-588	0402-w402	1055	20E
589-592	717-917	1056-1069	J(a,c)
593-639	Q(c,b)	1070	11E
640	0802	1071-1073	21E-41E
641-703	C(a,d)	1074-1083	07111-97111
704	0w02	1084-1089	5072-p072
705	1w02	1090-1092	2404-4404
706-814	C(b,d)-H(c)	1093-1097	1043-5043
815-818	50p-80p	1098-1104	3172-9172
819-833	I(a,b,d)	1105-1119	G(c,b)
834	08101	1120	0604
835	18101	1121-1134	L(a,b,c)
836-879	J(c,b)-K(b,c)	1135	10r
880	0703	1136	20r
881-943	C(c,d)	1137-1183	Q(c,d)
944	0q03	1184	0p04
945-958	L(c,d,a)	1185	1p04
959-961	p0401-w0401	1186-1199	J(b,c)
962-978	O(b,c,d)	1200	0q04
979-984	2813-7813	1201-1204	04021-34021

<u>integer</u>	<u>Representation</u>	<u>Integer</u>	<u>Representation</u>
1205-1248	$L(c,d,b)-R(c,b,d)$	1701-1713	005001-w05001
1249	07021	1714-1744	$R(b,c,e)$
1250-1374	$B(d)-A(e)$	1745-1754	071101-971101
1375	51901	1755-1759	2094-6094
1376	61901	1760	0E0101
1377-1439	$C(b,e)$	1761	1E0101
1440-1447	090001-790001	1762-1792	$P(c,d)$
1448-1457	00711-90711	1793-1804	1007-w007
1458-1504	$Q(b,e)$	1805-1807	p03101-w03101
1505	07031	1808-1854	$Q(c,e)$
1506-1520	00012-r0012	1855	20w11
1521	091001	1856-1862	0407-6407
1522-1552	$R(d,c,a)$	1863-1874	007001-q07001
1552-1614	$C(c,e)$	1875-1983	$H(d)-C(d,e)$
1615	21w01	1984	0w07
1616-1629	$L(e,c,a)$	1985-1998	$L(d,e,a)$
1630-1632	41911-61911	1999-2001	q2103-e2103
1633-1663	$O(b,c,e)-G(c,d)$	2002-2018	$O(b,d,e)$
1664	0306	2019-2032	10032-r0032
1665-1678	$L(a,d,c)$	2033-2043	07001-p7001
1679-1682	10E01-40E01	2044-2047	30831-60831
1683-1685	093001-293001	2048-2059	0008-q008
1686-1692	07402-67402	2060-2063	00p02-30p02
1693-1698	20p11-70p11	2064-2094	$K(e,c)$
1699	07112	2095	90422
1700	17112	2096-2104	0308-8308

<u>Integer</u>	<u>Representation</u>	<u>Integer</u>	<u>Representation</u>
2105-2112	02712-72712	2289-2298	070111-970111
2113-2119	1408-7408	2299-2302	05912-34912
2120-2129	107101-p07101	2303	00E02
2130	07032	2304-2314	0009-p009
2131-2161	K(c,d)	2315-2318	21841-51841
2162	0q151	2319	01E02
2163	1q151	2320-2334	000401-r00401
2164-2176	003011-w03011	2335-2338	905011-w05011
2177-2193	O(c,d,e)	2339-2353	002111-r02111
2194-2199	1418-6418	2354-2359	1309-7309
2200-2208	008101-808101	2360-2364	1077-5077
2209-2223	020111-r20111	2365-2368	01E001-31E001
2224	0p301	2369-2374	1409-6409
2225	1p301	2375-2384	007201-707201
2226-2234	0128-8128	2385-2394	0019-9019
2235-2242	00912-70912	2395-2399	417201-817201
2243	0p2011	2400	05041
2244	1p2011	2401-2479	A(f)
2245-2258	004011-E04011	2480-2481	20w12-30w12
2259	08013	2482-2544	C(b,f)
2260	18013	2545	0900001
2261-2273	014011-w14011	2546-2654	Q(d,e)-B(e)
2274-2287	000042-E0042	2655	509011
2288	0r0301	2656	609011

<u>Integer</u>	<u>Representation</u>	<u>Integer</u>	<u>Representation</u>
2657-2719	C(c,f)	3136-3149	020202-E20202
2720	080002	3150-3151	20804-30804
2721-2734	L(c,f,a)	3152-3157	030202-530202
2735-2737	p040001-w040001	3158-3162	12005-52005
2738-2784	O(b,c,f)-P(b,e)	3163-3168	1290001-5290001
2785-2794	071002-971002	3169-3199	K(c,f)
2795-2799	72014-q2014	3200	070701
2800	w2014	3201-3209	021701-821701
2801	0901001	3210-3216	018021-618021
2802-2832	R(d,c,e)	3217-3263	Q(e,d)
2833-2836	100601-400601	3264-	w01031
2837-2894	L(b,c,d)-Q(e,c)	3265	E01031
2895-2899	8060001-w060001	3266-3279	002202-E02202
2900-2912	0031001-w031001	3280	q030101
2913-2959	Q(c,f)	3281	w030101
2960-2967	070102-770102	3282-3328	O(c,d,f)-R(e,d,b)
2968-2979	0070001-q070001	3329-3338	070012-970012
2980	012121	3339-3345	01714-61714
2981-3024	L(c,b,f)-R(c,b,f)	3346-3349	0401101-3401101
3025	0702001	3350-3363	L(d,f,b)
3026-3088	C(d,f)	3364-3368	0q20101-4q20101
3089	Op1102	3369-3377	007121-807121
3090-3103	L(d,f,a)	3378	0601101
3104-3134	P(c,e)	3379-3393	I(b,e,d)
3135-	p0005	3394-3403	0701101-9701101

<u>Integer</u>	<u>Representation</u>	<u>Integer</u>	<u>Representation</u>
3404-3409	00814-50814	3604-3608	102q1-502q1
3410-3418	071012-871012	3609-3618	0170101-9170101
3419	06714	3619-3633	M(b,d,f,c)
3420	16714	3634-3637	05092-35092
3421-3424	04w001-33w001	3638-3641	0241101-3241101
3425-3439	G(c,f)	3642-3649	0092001-7092001
3440	060801	3650	0702101
3441-3454	L(a,f,c)	3651-3759	Q(d,f)-C(e,f)
3455-3459	10E0001-50E0001	3760	090402
3460-3467	00282-70282	3761-3774	L(e,f,a)
3468-3472	10p1001-50p1001	3775-3777	10w121-30w121
3473-3503	R(e,d,c)	3778-3794	O(b,e,f)
3504	02w04	3795-3803	101711-901711
3505	12w04	3804-3808	00q2001-40q2001
3506-3519	L(b,f,c)	3809	0700011
3520	20w021	3810-3824	M(b,e,f,a)
3521	30w021	3825-3834	0800011-9800011
3522-3527	0114001-5114001	3835-3840	00734-50734
3528-3533	017411-517411	3841	0900011
3534-3537	01w021-31w021	3842-3872	P(d,e)
3538-3568	R(c,d,f)	3873-3879	0q00011-6q00011
3569-3573	070611-470611	3880	067012
3574-3579	07115-57115	3881-3887	0271101-6271101
3580-3585	10724-60724	3888-3934	H(e)
3586-3593	170112-870112	3935-3939	00p05-40p05
3594-3603	1070101-p070101	3940-3952	0030011-w030011

<u>Integer</u>	<u>Representation</u>	<u>Integer</u>	<u>Representation</u>
3953-3969	O(c,e,f)	4045-4046	04p03l-14p03l
3970-3978	100811-900811	4047-4049	00r200l-20r200l
3979-3982	10763-30763	4050-4051	011E1-111E1
3982-3984	00r102-20r102	4052-4065	L(c,e,d)
3985-3999	M(c,e,f,a)	4066-4074	1701011-9701011
4000-4009	070003-970003	4075-4078	0491101-3491101
4010	00644	4079-4083	00E0101-40E0101
4011-4017	0091101-6091101	4084-4088	020141-420141
4018-4020	0016001-2016001	4089-4094	0184001-5184001
4021-4034	L(e,f,b)	4095	01E0101
4035-4042	071022-771022	4096-4174	A(g)
4043-4044	0291101-1291101		

VITA

Fred Winchell Sparks was born at Georgetown, Texas on November 13, 1891. He was graduated from the High School at Lampasas, Texas in 1908, and he received the A. B. and the A. M. degrees from Southwestern University in the years 1920 and 1922, respectively. In the summer of 1922 he entered the University of Chicago, from which he received the S. M. degree in June, 1923. Since receiving the above degree, he has attended the University of Texas one quarter and the University of Chicago six quarters.

His teaching experience comprises five years in the public schools of Texas, three years at Southwestern University, one year at the Agricultural and Mechanical College of Texas, two years at the Louisiana State Normal College, and three years at Texas Technological College.

At the the University of Chicago he studied under Professors Slaught, Young, Moulton, MacMillan, Bliss, Logsdon, Bell, Mitchell, Graves, Barnard, and Dickson. He wishes especially to express his gratitude to Professor Dickson, under whose direction this dissertation was prepared.

